# Stability of Two Superposed Viscoelastic (Walters B′) Fluid-Particle Mixtures in Porous Medium

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The stability of the plane interface separating two viscoelastic (Walters B′) superposed fluids in porous medium in the presence of suspended particles is considered. For the case of two uniform Walters B′ fluids separated by a horizontal boundary, the system is found to be stable or unstable under certain conditions for the stable configuration. However, the system is found to be unstable for the unstable configuration. The case of an exponentially varying density is also considered. For the stable stratification, the system is found to be stable or unstable under certain conditions, whereas the system is found to be unstable for the unstable stratification. The behaviour of growth rates with respect to suspended particle, number density, and medium permeability is examined analytically.

## Introduction

The instability derived from the character of the equilibrium of an incompressible heavy fluid of variable density (i.e. of a heterogeneous fluid) is termed as the Rayleigh-Taylor instability. Mention may be made of two important special cases: (a) two fluids of different densities superposed one over the other; (b) a fluid with a continuous density stratification. Chandrasekhar [1] has given a detailed account of the Rayleigh-Taylor instability in non-porous medium.

The flow through porous media is of considerable interest amongst petroleum engineers and geophysical fluid dynamicists. Lapwood [2] has studied the stability of convective flow in hydrodynamics in a porous medium using the Rayleigh procedure. Wooding [3] has considered the Rayleigh instability of a thermal boundary layer in flow through porous medium. In geophysical situations, more often than not the fluid is not pure but may instead be permeated with suspended (or dust) particles. Scanlon and Segel [4] have considered the effect of suspended particles on the onset of Bénard convection and found that the critical Rayleigh number was reduced solely because the heat capacity of the pure gas was supplemented by that of the particles. The thermal instability of fluids in a porous medium in the presence of suspended particles has been studied by Sharma and Sharma [5]. The suspended particles and the permeability of the medium were found to destabilize the layer. The gross effect, when the fluid slowly percolates the pores of the rock, is represented by the well known Darcy's law. The fluid has been considered to be Newtonian in all the above studies.

With the growing importance of non-Newtonian fluids in modern technology and industries, investigations on such fluids are desirable. Oldroyd [6] proposed a theoretical model for a class of viscoelastic fluids. An experimental demonstration by Tom and Strawbridge [7] reveals that a dilute solution of methyl methacrylate in *n*-butyl acetate agrees well with the theoretical model of the Oldroyd fluid. Slattery [8] has studied the flow of viscoelastic fluids through a porous medium. Kumar [9] has studied the Rayleigh-Taylor instability of a Newtonian viscous fluid overlying an Oldroydian viscoelastic fluid containing suspended particles in a porous medium. In another study, Sharma and Kumar [10] have studied the thermal instability of an Oldroydian viscoelastic fluid in a porous medium and have also considered the effect of uniform rotation on the instability.

There are many elastico-viscous fluids that cannot be characterized by Oldroyd's [6] constitutive relations. One such class of elastico-viscous fluids is Walters B′ fluid. Sharma and Kumar [11] have studied the stability of the plane interface separating two viscoelastic (Walters B′) superposed fluids of uniform densities. It is this class of elastico-viscous fluids we are interested in particularly in the study of the stability of two superposed Walters B′ viscoelastic fluids permeated with suspended particles in porous medium. The knowledge regarding viscoelastic fluid-particle mixtures is not commensurate with their scientific and industrial importance. The analysis would be relevant to the stability of

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some polymer solutions, and the problem finds its usefulness in several geophysical situations and in chemical technology. These aspects have motivated the present study.

## Formulation of the Problem and Perturbation Equations

Consider an incompressible viscoelastic (Walters B′) fluid-particle layer consisting of a viscoelastic fluid of density $\rho$, permeated with suspended particles of density $mN$, arranged in horizontal strata in a porous medium. The system is acted on by the gravity force $\boldsymbol{g}(0, 0, -g)$. Let $p$, $\rho$, $\mu$, $\mu'$ and $\boldsymbol{v}(u, v, w)$ denote respectively the pressure, density, viscosity, viscoelasticity and velocity of the pure fluid; $\boldsymbol{u}(l, r, s)$, $m$ and $N(\bar{x}, t)$ denote the velocity, mass and number density of the particles, respectively, $\varepsilon$ is the medium porosity, $k_1$ the medium permeability, and $\bar{x}=(x, y, z)$. The equations of motion and continuity for the fluid are

$$\frac{\rho}{\varepsilon}\left[\frac{\partial \boldsymbol{v}}{\partial t}+\frac{1}{\varepsilon}(\boldsymbol{v}\cdot\nabla)\boldsymbol{v}\right]= -\nabla p+\rho\boldsymbol{g}-\frac{1}{k_1}\left(\mu-\mu'\frac{\partial}{\partial t}\right)\boldsymbol{v} +\frac{KN}{\varepsilon}(\boldsymbol{u}-\boldsymbol{v}), \quad (1)$$

$$\nabla\cdot\boldsymbol{v}=0, \quad (2)$$

where $K=6\pi\mu\eta$ is the Stokes' drag coefficient ($\eta$ being the particle radius).

Since the density of every fluid particle remains unchanged as we follow it with its motion, we have

$$\varepsilon\frac{\partial\rho}{\partial t}+(\boldsymbol{v}\cdot\nabla)\rho=0, \quad (3)$$

Assuming uniform particle size, spherical shape and small relative velocities between the fluid and particles, the presence of particles adds an extra force term in the equation of motion (1), proportional to the velocity difference between particles and fluid. The force exerted by the fluid on the particles is equal and opposite to that exerted by the particles on the fluid. Interparticle reactions are ignored since we assume that the distances between the particles are quite large compared with their diameter. The effects of pressure, gravity, and Darcian force on the suspended particles are negligibly small and therefore ignored. Under the above assumptions, the equations of motion and continuity for the particles are

$$mN\left[\frac{\partial \mathrm{u}}{\partial t}+\frac{1}{\varepsilon}(\mathrm{u}\cdot\nabla)\right]=KN(\mathrm{v}-\mathrm{u}), \quad (4)$$

$$\varepsilon\frac{\partial N}{\partial t}+\nabla\cdot(N\boldsymbol{u})=0. \quad (5)$$

The initial state of the system is taken to be a quiescent layer (no settling) with a uniform particle distribution $N_0$, i.e. $\boldsymbol{v}=(0, 0, 0)$, $\boldsymbol{u}=(0, 0, 0)$, and $N=N_0$ is a constant. The character of the equilibrium of this initial static state can be determined, as usual, by supposing that the system is slightly disturbed and then following its further evolution.

Let $\partial\rho$, $\partial p$, $\boldsymbol{v}(u, v, w)$, and $\boldsymbol{u}(l, r, s)$ denote, respectively, the perturbations in density $\rho$, pressure $p$, velocity of fluid and velocity of particles. Then the linearized perturbed forms of (1)–(5) become

$$\frac{\rho}{\varepsilon}\frac{\partial\boldsymbol{v}}{\partial t}=-\nabla\delta p+\boldsymbol{g}\,\delta\rho-\frac{\rho}{k_1}\left(\nu-\nu'\frac{\partial}{\partial t}\right)\boldsymbol{v} +\frac{KN}{\varepsilon}(\boldsymbol{u}-\boldsymbol{v}), \quad (6)$$

$$\nabla\cdot\boldsymbol{v}=0, \quad (7)$$

$$\varepsilon\frac{\partial}{\partial t}(\delta\rho)=-w(D\rho), \quad (8)$$

$$\left(\frac{m}{K}\frac{\partial}{\partial t}+1\right)\boldsymbol{u}=\boldsymbol{v}, \quad (9)$$

and

$$\frac{\partial\rho}{\partial t}+(\boldsymbol{v}\cdot\nabla)\rho=0, \quad (10)$$

where $M=\varepsilon N/N_0$, and $N_0$, $N$, $\nu\,(=\mu/\rho)$, $\nu'\,(=\mu'/\rho)$ stand for initial uniform number density, perturbation in number density, kinematic viscosity, and kinematic viscoelasticity, respectively, and $D=\mathrm{d}/\mathrm{d}z$.

Analyzing the disturbances into normal modes, we seek solutions whose dependence on $x$, $y$, and $t$ is given by

$$\exp(ik_x x+ik_y y+nt), \quad (11)$$

where $k_x$, $k_y$ are horizontal wave numbers, $k^2=k_x^2+k_y^2$, and $n$ is a complex constant.

For a perturbation of the form (11), equations (6)–(9) after eliminating $\boldsymbol{u}$ give

$$\left[n'+\frac{\nu-\nu' n}{k_1}\right]\rho u=-ik_x\,\delta p, \quad (12)$$

$$\left[n' + \frac{\nu - \nu' n}{k_1}\right] \rho v = -i k_y \delta p, \quad (13)$$

$$\left[n' + \frac{\nu - \nu' n}{k_1}\right] \rho w = -D \delta p - g \delta \rho, \quad (14)$$

$$i k_x u + i k_y v + D w = 0, \quad (15)$$

$$\varepsilon n \delta \rho = -w D\rho, \quad (16)$$

where

$$n' = \frac{n}{\varepsilon}\left(1 + \frac{m N K/\rho}{m n + K}\right).$$

Eliminating $\delta p$ between (12)–(14) and using (15) and (16), we obtain

$$n[D(\rho Dw) - k^2 \rho w] + \frac{1}{k_1} \cdot [D(\rho \nu Dw) - k^2 \rho \nu w] - \frac{n}{k_1} \cdot [D(\rho \nu' Dw) - k^2 \rho \nu' w] = -\frac{g k^2}{\varepsilon n}(D\rho) w. \quad (17)$$

## Two Superposed Viscoelastic (Walters B′) Fluids Separated by a Horizontal Boundary

Considered the case when two superposed Walters B′ fluids of uniform densities $\rho_1$ and $\rho_2$, uniform viscosities $\mu_1$ and $\mu_2$, uniform viscoelasticities $\mu_1'$ and $\mu_2'$ are separated by a horizontal boundary at $z=0$. The subscripts 1 and 2 distinguish the upper and lower fluids, respectively. Then in each region of constant $\rho$, constant $\mu$ and constant $\mu'$, (17) reduces to

$$(D^2 - k^2) w = 0. \quad (18)$$

The general solution of (18) is

$$w = A e^{+kz} + B e^{-kz}, \quad (19)$$

where $A$ and $B$ are arbitrary constants.

The boundary conditions to be satisfied in the present problem are:

(i) The velocity should vanish when $z \to +\infty$ (for the upper fluid) and $z \to -\infty$ (for the lower fluid).
(ii) $w(z)$ is continuous at $z=0$.
(iii) The jump condition at the interface.

Applying the boundary conditions (*i*) and (*ii*), we have

$$w_1 = A e^{+kz}, \quad (z<0), \quad (20)$$

$$w_2 = A e^{-kz}, \quad (z>0), \quad (21)$$

where the same constant $A$ has been chosen to ensure the continuity of $w$ at $z=0$.

Equation (17) gives the jump condition at the interface $z=0$ as

$$n' \Delta_0 (\rho Dw) + \frac{1}{k_1} \Delta_0 (\rho \nu Dw) - \frac{n}{k_1} \Delta_0 (\rho \nu' Dw) + \frac{g k^2}{\varepsilon n} \Delta_0 (\rho) w_0 = 0. \quad (22)$$

Applying the condition (22) to the solutions (20) and (21), we obtain

$$\left[1 - \frac{\varepsilon}{k_1}(\alpha_2 \nu_2' + \alpha_1 \nu_1')\right] n^3 + \left[\frac{K}{m} + \frac{2NK}{\rho_1 + \rho_2} + \frac{\varepsilon}{k_1}(\alpha_2 \nu_2 + \alpha_1 \nu_1) - \frac{K\varepsilon}{m k_1}(\alpha_2 \nu_2' + \alpha_1 \nu_1')\right] n^2 + \left[\frac{K\varepsilon}{m k_1}(\alpha_2 \nu_2 + \alpha_1 \nu_1) + g k (\alpha_1 - \alpha_2)\right] n + \frac{K g k}{m}(\alpha_1 - \alpha_2) = 0, \quad (23)$$

where

$$\alpha_{1,2} = \frac{\rho_{1,2}}{\rho_1 + \rho_2}, \quad \nu_{1,2} = \frac{\mu_{1,2}}{\rho_{1,2}}, \quad \nu_{1,2}' = \frac{\mu_{1,2}'}{\rho_{1,2}}.$$

*(a) Stable Case* ($\alpha_2 < \alpha_1$)

We rewrite (23) in the form

$$n^3 + B n^2 + C n + D = 0, \quad (23')$$

where $B$, $C$, and $D$ are the coefficients of $n^2$, $n$ and the constant term after dividing by the coefficient of $n^3$ throughout in (23). It is clear from (23′) that $D>0$, and it has been found, after calculations, that for

$$1 > \frac{\varepsilon}{k_1}(\alpha_2 \nu_2' + \alpha_1 \nu_1') \quad (24)$$

$B C - D > 0$. Since $D>0$ and $B C - D > 0$, (Chandrasekhar [1], p. 463], the real parts of all roots of (23) are negative. The system is therefore stable. But if

$$1<\frac{\varepsilon}{k_1}(\alpha_2\, v_2'+\alpha_1\, v_1'), \tag{25}$$

the coefficient of $n^3$ in (23) is negative. Equation (23) therefore allows one change of sign and so has one positive root. The occurrence of a positive root implies that the system is unstable.

*(b) Unstable Case ($\alpha_2>\alpha_1$)*

For a potentially unstable case, the constant term in (23) is negative. Equation (23) therefore allows one change of sign and so has one positive root, and hence the system is unstable.

## The Case of Exponentially Varying Density

Here we consider density stratification in a fluid of depth $d$ as

$$\rho(z)=\rho_0\, e^{\beta z}, \tag{26}$$

where $\rho_0$ and $\beta$ are constants. Assume that $\beta d \ll 1$, i.e. the variation of the density at two neighbouring points in the velocity field, which is much less than the average density, has a negligible effect on the inertia of the fluid.

The boundary conditions for the case of two free surfaces are

$$w=0,\quad D^2 w=0 \quad \text{at} \quad z=0 \quad \text{and} \quad z=d. \tag{27}$$

The proper solution of (17), satisfying (27) is

$$w=w_0 \sin\frac{s\pi z}{d}, \tag{28}$$

where $w_0$ is a constant and $s$ is an integer.

Substituting (28) in (17) and neglecting the effects of heterogeneity of the inertia, we get

$$\left[1-\frac{\varepsilon v'}{k_1}\right]n^3+\left[\frac{K}{m}\left(1-\frac{\varepsilon v'}{k_1}\right)+\frac{KN}{\rho}+\frac{\varepsilon v}{k_1}\right]n^2 + \left[\frac{\varepsilon v k}{m k_1}-\frac{g\beta k^2}{(s\pi/d)^2+k^2}\right]n-\frac{g\beta k^2 K/m}{(s\pi/d)^2+k^2}=0. \tag{29}$$

### Stable Stratification ($\beta<0$)

Re-write (29) in the form

$$n^3+B' n^2+C' n+D'=0, \tag{29'}$$

where $B'$, $C'$, and $D'$ are the coefficients of $n^2$, $n$ and the constant term after dividing by the coefficient of $n^3$ throughout in (29). It is clear from (29′) that $D'>0$, and it has been found "after calculations" that for

$$1>\frac{\varepsilon v'}{k_1} \tag{30}$$

$B'C'-D'>0$. Since $D'>0$ and $B'C'-D'>0$, (Chandrasekhar [1], p. 463), the real parts of all roots of (29′) are negative. The system is therefore stable.

But if

$$1<\frac{\varepsilon v'}{k_1}, \tag{31}$$

the coefficient of $n^3$ in (29) is negative. Equation (29) therefore allows one change of sign and so has one positive root. The occurrence of this positive root implies that the system is unstable.

### Unstable Stratification ($\beta>0$)

For the unstable stratifications ($\beta>0$), the constant term in (29) is negative. Equation (29) therefore allows one change of sign and so has one positive root and hence the system is unstable.

Let $n_0$ denote the positive root of (29). Then

$$\left[1-\frac{\varepsilon v'}{k_1}\right]n_0^3+\left[\frac{K}{m}\left(1-\frac{\varepsilon v'}{k_1}\right)+\frac{KN}{\rho}+\frac{\varepsilon v}{k_1}\right]n_0^2 + \left[\frac{\varepsilon v k}{m k_1}-\frac{g\beta k^2}{(s\pi/d)^2+k^2}\right]n_0-\frac{g\beta k^2 K/m}{(s\pi/d)^2+k^2}=0. \tag{32}$$

To find the role of the particle number density and medium permeability concerning the growth rate of unstable modes, we examine the natures of $\mathrm{d}n_0/\mathrm{d}N$ and $\mathrm{d}n_0/\mathrm{d}k_1$ analytically. Equation (32) yields

$$\frac{\mathrm{d}n_0}{\mathrm{d}N}=\frac{(K/\rho)\,n_0^2}{\dfrac{g\beta k^2}{(s\pi/d)^2+k^2}-\left[3n_0^2\left(1-\dfrac{\varepsilon v'}{k_1}\right)+2n_0\left\{\dfrac{K}{m}\left(1-\dfrac{\varepsilon v'}{k_1}\right)+\dfrac{KN}{\rho}+\dfrac{\varepsilon v}{k_1}\right\}+\dfrac{\varepsilon v K}{m k_1}\right]} \tag{33}$$

and

$$\frac{\mathrm{d}n_0}{\mathrm{d}k_1} = \frac{\left[-\frac{e\,n'}{k_1^2}\right]n_0^3 + \left[\frac{e\,n}{k_1} - \frac{e\,n'\,K}{m\,k_1^2}\right]n_0^2 + \left[\frac{e\,n\,K}{m\,k_1^2}\right]n_0}{3\,n_0^2\left[1 - \frac{e\,n'}{k_1}\right] + 2\,n_0\left[\frac{K}{m}\left(1 - \frac{e\,n'}{k_1}\right) + \frac{K\,N}{r} + \frac{e\,n}{k_1}\right] + \left[\frac{e\,n\,K}{m\,k_1} - \frac{g\,b\,k^2}{(s\,\pi\,/\,d)^2 + k^2}\right]}\,. \tag{34}$$

It is clear from (33) and (34) that $\mathrm{d}n_0/\mathrm{d}N$ and $\mathrm{d}n_0/\mathrm{d}k_1$ may be positive or negative. The growth rates, thus, both decrease (for certain wave numbers) and increase (for other wave numbers) with the increase in particle number density and medium permeability for the unstable stratifications.


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